



# Fermi National Accelerator Laboratory

Fermilab-Conf-83/45-THY  
May, 1983

## A NEW SOLUBLE APPROXIMATION TO FOKKER-PLANCK EQUATIONS

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Talk delivered at the 49th semi-annual Statistical Mechanics Meeting, May 12-13, 1983, Rutgers University.

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The results that I am about to discuss were obtained in the course of work on collision rates in electron-positron colliding-beam storage rings.<sup>1</sup> Understanding collision rates requires understanding the statistical distribution of beam particles.

Idealizing and simplifying for the sake of clarity, an electron circles a storage ring in the horizontal plane at the speed of light, with its displacement  $y$  perpendicular to the plane of the ring executing small oscillations according to

$$\ddot{y} + \gamma(t)\dot{y} - F(y, t) = \eta(t) \xi(t), \quad (1)$$

where  $\gamma$ ,  $\eta$ , and  $F$  are modulated periodically in  $t$  (the period is the time needed to circle the ring once), and  $\xi$  is Gaussian uncorrelated white noise

$$\langle \xi(t) \xi(t') \rangle = \delta(t - t'). \quad (2)$$

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Correspondingly, the probability distribution,  $P$ , in  $y$  and  $\dot{y}$  evolves in time according to a Fokker-Planck equation

$$\frac{\partial P}{\partial t} + \dot{y} \frac{\partial P}{\partial y} + F \frac{\partial P}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} (\gamma \dot{y} P) + \frac{1}{2} \eta^2 \frac{\partial^2 P}{\partial \dot{y}^2}. \quad (3)$$

The Langevin equation (1), or the Fokker-Planck equation (3), differs from textbook systems primarily in that the modulation period is much smaller, rather than much larger, than the naive thermal relaxation time,  $1/\langle \gamma \rangle$  (the averaging is over external modulation). In my work, I am particularly concerned with the limit  $\gamma \rightarrow 0, \eta \rightarrow 0$ , with  $F$  and the modulation period fixed.

For today's talk, I shall ignore <sup>(of parameters)</sup> modulation. The new results described here, in the present one-dimensional context, constitute a necessary prerequisite for more realistic studies. I will report soon on a more extensive study of modulation-related statistical effects (and enlarge on the topics treated in the present talk) in a forthcoming series of publications.

The analysis I describe today concerns the "semiclassical" (small -  $\eta$ ) approximation to the solution of (3) corresponding to the probability that a Langevin particle, initially at  $y=y_1$  and  $\dot{y}=v_1$ , ends up at  $y_2$  and  $v_2$  at time  $t$ :

$$P(y_1, v_1; y_2, v_2; t) \cong N \exp \left\{ \frac{-1}{2\eta^2} \int_0^t d\tau \left( [\ddot{y} + \gamma \dot{y} + u'(y)]^2 \equiv L(y, \dot{y}, \ddot{y}) \right) \right\} \quad (4)$$

where  $N$  is a normalization constant. (This is most easily derived from path integral considerations.<sup>2</sup>) I have replaced the force  $F$  by minus the derivative of a (confining and symmetric, for definiteness) potential  $U$ . The path  $y(\tau)$  extremizes the exponent, subject to the boundary conditions indicated in the parentheses on the left-hand side. The extremal ("Onsager-Machlup") equation is

$$0 = \left[ \frac{d^2}{d\tau^2} - \gamma \frac{d}{d\tau} + U''(y) \right] \left[ \ddot{y} + \gamma \dot{y} + U'(y) \right]. \quad (5)$$

I am concerned with solutions of (5) and (4) for small  $\gamma$  but for large time ( $t \gg O(1/\gamma)$ ), because in storage rings,  $1/\langle \gamma \rangle$  = milliseconds, much shorter than the time scale (minutes, hours) on which elementary-particle experiments are performed.

In order to solve (5) approximately for small  $\gamma$ , I assume that at any time,  $y$  is close to a solution of frictionless, noiseless Newton's law,  $\ddot{y} + U'(y) = 0$ , up to a deviation that vanishes as  $\gamma \rightarrow 0$  ( $t\gamma$  fixed). I then proceed as follows:

I take the interval  $(0, t)$  and break it into subintervals - a subinterval (except for the first and last) begins and ends at two successive times when  $y$  crosses zero (where  $U' = 0$ ). Hardly any damping or fluctuation takes place in one such subinterval. The deviation of  $y$  in such a subinterval from Newtonian vibration depends on  $\gamma$ , and also on the way initial acceleration deviates from  $-U'(0) = 0$ , and also

on how the initial  $\ddot{y}$  differs from the Newtonian value  $-U''(0)\dot{y}$ .

I compute the solution  $y$  to (5) in a subinterval by perturbing about a solution to undamped Newtonian motion in powers of  $\gamma$ , and of  $a$  (acceleration (times sign of initial velocity) at beginning of subinterval), and of  $p$  (value of  $\ddot{y} + U''\dot{y}$  (times sign of initial velocity) at beginning of subinterval). From this computation I derive an approximate map, which transforms the values of  $a$ ,  $p$ , and speed  $v$  at the beginning of one subinterval into their values at the beginning of the next.

This one-subinterval map is to be iterated many ( $\sim O(1/\gamma)$ ) times. My main result is that such long-time iteration can be done in a simple closed form, up to small remainders.

A technical note: It is necessary to assign formal orders in  $\gamma$  to the variables  $a$  and  $p$  in order to make the perturbative calculation systematic. I take my cue from two particular solutions to (5),

$$\ddot{y} \pm \gamma \dot{y} + U'(y) = 0 \quad (6)$$

(damped and antidamped Newtonian motion). When  $y=0$ , (6) gives  $a = \mp \gamma v$ , and  $p = \gamma^2 v$ . So in the general case I take  $v = O(1)$ ,  $a = O(\gamma)$ , and  $p = O(\gamma^2)$ .

Then the perturbative subinterval map is

$$\Delta v = a \left( \frac{J}{v^2} \right) + O(\gamma^2), \quad (7a)$$

$$\Delta a = \frac{\gamma^2 v}{2} \frac{d^2 J}{dv^2} - p v \frac{d}{dv} \left( \frac{1}{v} \frac{dJ}{dv} \right) + \frac{a^2}{2} \frac{d^2}{dv^2} \left( \frac{J}{v} \right) + O(\gamma^3),$$

$$\begin{aligned} \Delta p = \gamma^2 a \left( \frac{J}{v^2} + \frac{1}{2} \frac{d^2 J}{dv^2} \right) - a p \left( \frac{J}{v^3} + \frac{d}{dv} \left( \frac{1}{v} \frac{dJ}{dv} \right) \right) \\ + \frac{a^3}{2v} \frac{d^2}{dv^2} \left( \frac{J}{v} \right) + O(\gamma^4), \end{aligned} \quad (7b)$$

where  $J$  is  $\pi$  times the canonical action integral for undamped Newtonian motion at energy  $v^2/2$  ( $J \equiv \frac{1}{2} \oint \dot{y}^2 d\tau$ ). Such an undamped Newtonian oscillation completes itself in time  $(2/v)dJ/dv$ .

For small  $\gamma$ , we may, because of (7), consider  $\Delta v$ ,  $\Delta a$ ,  $\Delta p$  as differentials rather than finite differences. Then (7) is a system of three first-order ODE's. This system turns out to have two conserved quantities

$$\begin{aligned} E &\equiv -v^2 \gamma^2 - a^2 + 2pv, \\ H &\equiv \frac{1}{2} \left[ \frac{E}{v} \frac{dJ}{dv} - \gamma^2 J + a^2 \left( \frac{J}{v^2} \right) \right]. \end{aligned} \quad (8)$$

The former,  $E$ , is the restriction to  $y=0$  of

$$E = L - \dot{y} \frac{\partial L}{\partial \dot{y}} - \ddot{y} \frac{\partial L}{\partial \ddot{y}} + \dot{y} \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{y}} \right), \quad (9)$$

which is actually conserved exactly by (5), as a consequence of our present neglect of any explicit time-dependence of parameters. The latter,  $H$ , is a kind of adiabatic invariant,

$$H = \frac{1}{2} \oint \left[ L - \dot{y} \frac{\partial L}{\partial \dot{y}} - \ddot{y} \frac{\partial L}{\partial \ddot{y}} + \frac{1}{2} \frac{d}{d\tau} \left( \dot{y} \frac{\partial L}{\partial \dot{y}} \right) \right] d\tau. \quad (10)$$

When the right-hand sides of (7) are written in terms of  $H$ , the equations for  $\Delta v$  and  $\Delta a$  take a canonical Hamiltonian form

$$\Delta v = \frac{\partial H(v, a, E)}{\partial a} + O(\gamma^2),$$

$$\Delta a = - \frac{\partial H(v, a, E)}{\partial v} + O(\gamma^3). \quad (11)$$

Because of these conservation laws, the system (7) can be solved (approximately) by quadrature. The end result is this: In terms of  $E$  and  $H$ , the probability in equation (4) (restricted, for simplicity, to  $y_1=y_2=0$ ) is

$$P(0, v_1; 0, v_2; t) \cong N \exp \frac{-1}{2\eta^2} \left\{ \gamma(v_2^2 - v_1^2) + \int_{|v_1|}^{|v_2|} \frac{[2H - \frac{E}{v} \frac{dJ}{dv} + 2\gamma^2 J]}{[J(2H - \frac{E}{v} \frac{dJ}{dv} + \gamma^2 J)]^{3/2}} v |dv| + O(\gamma^2) \right\} \quad (12)$$

The path of integration proceeds from  $|v_1|$  down to a turning point, and then up to  $|v_2|$ . To determine  $E$  and  $H$ , one extremizes the exponent in (12) (with  $v_1$  and  $v_2$  fixed) subject to the constraint

$$\gamma t = \gamma \sum \left( \frac{1}{v} \frac{dJ}{dv} + O(\gamma) \right)$$

$$= \gamma \int_{|v_1|}^{|v_2|} \frac{|dv| (dJ/dv)}{[J(2H - \frac{E}{v} \frac{dJ}{dv} + \gamma^2 J)]^{3/2}} + O(\gamma). \quad (13)$$

Here is a check: As  $\gamma t \rightarrow \infty$ , it turns out that both  $E$  and  $H$  approach zero. Then the probability (12) becomes

$$N \exp \frac{-2\gamma}{\eta^2} \left( \frac{v_2^2}{2} \right), \quad (14)$$

i.e. Maxwell-Boltzmann, as it should. Incidentally,  $E=H=0$

means that  $v$ ,  $a$ , and  $p$  correspond to the restriction of solutions of equations (6) to  $y=0$ . This is to be interpreted as follows: When  $t \rightarrow \infty$ , the extremal path proceeds from  $(y_1, v_1)$  to  $(0,0)$  according to damped Newtonian motion, and then proceeds from  $(0,0)$  to  $(y_2, v_2)$  according to antidamped Newtonian motion.

Note that the harmonic oscillator is singular as far as this formalism is concerned. When  $U$  is quadratic in  $y$ , (12) and (13) depend on the same one linear combination of  $E$  and  $H$ , so that the constants cannot be determined independently. In this case, the formalism must be modified in a well-defined way. With the appropriate modifications, one can reproduce the exact<sup>3</sup> harmonic oscillator Fokker-Planck Green's function (to within  $O(\hbar^2)$  in the exponent).

#### ACKNOWLEDGEMENTS

I am grateful to Loyal Durand and Cosmas Zachos for very helpful conversations.

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